Abelian Categories towards Homological Algebra

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Definition

A category $\mathcal{C}$ consists of:

- Objects $\mathrm{obj}(\mathcal{C})$,
- Morphism sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for every object pair,
- Identity morphisms id_A for each object A,
- Composition functions $\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$.

We denote morphisms as $f : A \to B$ and use gf or $g \circ f$ for composition. Two axioms govern these: Associativity and Unit.

- $(hg)f = h(gf)$ for $f : A \to B$, $g : B \to C$, $h : C \to D$.
- $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$ for $f : A \to B$.

Example

Sets consists of sets as objects and set functions as morphisms. The morphisms from A to B are functions from A to B, and composition is function composition. Identity morphisms are functions $\mathrm{id}_A(a) = a$ for all $a \in A$.

Example

In **Ab**, objects are abelian groups and morphisms are group homomorphisms. Composition is ordinary composition of homomorphisms.

Example

- **Groups** is the category of groups and group maps,
- **Rings** is the category of rings and ring maps,
- $R-\mathbf{mod}$ is the category of left R-modules, where objects are left R-modules, morphisms are R-module homomorphisms, and composition is the usual composition.

Definition

An isomorphism $f : B \to C$ in $\mathcal{C}$ has an inverse $g : C \to B$ such that $gf = \mathrm{id}_B$ and $fg = \mathrm{id}_C$.

Example

In **Sets**, an isomorphism is a set bijection.

Example

In **Top** of topological spaces and continuous maps, an isomorphism is a homeomorphism.

Example

In the category of smooth manifolds and smooth maps, an isomorphism is called a diffeomorphism.

Definition

A morphism $f : A \to B$ is monic in $\mathcal{C}$ if $fg_1 = fg_2$ implies $g_1 = g_2$ for any distinct $g_1, g_2 : X \to A$.

$$X \underset{g_2}{\overset{g_1}{\rightrightarrows}} A \xrightarrow[f]{} B$$

Example

In concrete categories like **Sets** and **Ab**, monic morphisms are set injections.

Definition

A morphism $f : B \to C$ is epi in $\mathcal{C}$ if $g_1 f = g_2 f$ implies $g_1 = g_2$ for any distinct $g_1, g_2 : C \to D$.

$$\begin{tikzcd} B \arrow[r, "f"] & C \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & D \end{tikzcd}$$

Example

In categories like **Sets** and **Ab**, epis are surjective maps.

Example

In other concrete categories such as **Ring** or **Top** this fails; the morphisms whose underlying set map is onto are epi, but there are other epis.

- $\mathbb{Q} \hookrightarrow \mathbb{R}$
- $x \mapsto e^{ix}$ from $\mathbb{R} \mapsto S^1$

Definition

An initial object in $\mathcal{C}$, if it exists, is an object I with a unique morphism to any other object C.

Definition

A terminal object in $\mathcal{C}$, if it exists, is an object T with a unique morphism from any other object C.

Definition

An object that is both initial and terminal is called a zero object.

Example

In **Sets**, $\emptyset$ is the initial object, and any 1 element set is a terminal object. There is no zero object in **Sets**.

Example

In the category of rings **Ring**, where morphisms preserve unity, the ring of integers $\mathbb{Z}$ is an initial object. The zero ring consisting only of a single element $0 = 1$ is a terminal object.

Example

In the category of groups **Grp**, the trivial group (containing only the identity element) is both the initial and terminal object.

Definition

If $\mathcal{C}$ has a zero object 0, then every set $\mathrm{Hom}_{\mathcal{C}}(B, C)$ contains a distinguished element, denoted by 0, which is the composite $B \to 0 \to C$.

Definition

A kernel of a morphism $f : B \to C$ is a morphism $i : A \to B$ such that $fi = 0$ and every morphism $e : A' \to B$ in $\mathcal{C}$ with $fe = 0$ factors uniquely through A as $e = ie'$.

$$
\begin{tikzcd}
 & B \arrow[dr, "f"] & \\
 & A \arrow[u, "i"] \arrow[r, "0_{AC}"] & C \\
A' \arrow[uur, "e"] \arrow[ur, dashed, "e'"] \arrow[urr, "0_{A'C}"'] & &
\end{tikzcd}
$$

Example

Every kernel is monic, and any two kernels of f are isomorphic; often identified with the corresponding subobject of B.

Definition

Similarly, a cokernel of a morphism $f : B \to C$ is a morphism $p : C \to D$ such that $pf = 0$ and every morphism $g : C \to D'$ with $gf = 0$ factors uniquely through D as $g = g'p$ for a unique $g' : D \to D'$.

Example

Every cokernel is an epi, and any two cokernels are isomorphic. In **Ab** and R-**mod**, kernel and cokernel have their usual meanings.

$$
\begin{tikzcd}
 & C \arrow[d, "p"] \arrow[ddr, "g"] & \\
B \arrow[ur, "f"] \arrow[r, "0_{BD}"] \arrow[drr, "0_{BD'}"'] & D \arrow[dr, dashed, "g'"] & \\
 & & D'
\end{tikzcd}
$$

Definition

An **Ab**-category (preadditive) has every hom-set given abelian group structure, with composition distributing over addition.

$$A \xrightarrow{f} B \underset{g'}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$$

$h(g+g')f = hgf + hg'f$ in $\mathrm{Hom}(A, D)$

Definition

An additive category is an **Ab**-category with a zero object and products for every object pair.

Example

Finite products and coproducts coincide, often denoted $A \oplus B$.

Definition

An abelian category is an additive category with kernels, cokernels, and specific properties regarding monics and epis.

Example

In an abelian category, monics correspond to kernels, and epis to cokernels.

Example

The prototype abelian category is the category mod-R of R-modules, where R is a ring.

First Isomorphism Theorem

In any abelian category, the image $\mathrm{im}(f)$ of a map $f : B \to C$ is the subobject $\ker(\mathrm{coker}(f))$ of C.

Factorization of Maps

Every map $f : B \to C$ factors as:

$$B \xrightarrow{e} \mathrm{im}(f) \xrightarrow{m} C$$

where e is an epimorphism and m is a monomorphism.

Definition

A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of maps in an abelian category is called exact at B if $\ker(g) = \operatorname{im}(f)$.

Definition

A chain complex C of R-modules consists of a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules, along with R-module maps $d_n : C_n \to C_{n-1}$ such that $d \circ d = 0$, where d denotes the differential maps.

- The kernel of d_n is denoted $Z_n(C)$, representing the module of n-cycles.
- The image of d_{n+1} is denoted $B_n(C)$, representing the module of n-boundaries.
- Thus, $H_n(C) = Z_n(C)/B_n(C)$ gives the nth homology module of C.

Definition

There is a category **Ch**(mod-R) of chain complexes of (right) R-modules. Objects are chain complexes, and morphisms $u : C \to D$ are chain complex maps—families of R-module homomorphisms $u_n : C_n \to D_n$ commuting with d such that $u_{n-1} \circ d_n = d_n \circ u_n$.

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots \\
 & & \downarrow u & & \downarrow u & & \downarrow u & & \\
\cdots & \xrightarrow{d} & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \xrightarrow{d} & \cdots
\end{array}
$$

Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of chain complexes. Then there are natural maps $\partial : H^n(C) \to H^{n-1}(A)$, called connecting homomorphisms, such that the sequence

$$\cdots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(C) \longrightarrow \cdots$$

is exact.

Lemma

Snake – Consider a commutative diagram of R-modules of the form

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"'] & B' \arrow[r] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C & \end{tikzcd}$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker(f) \arrow[r, dotted] \arrow[d] & \ker(g) \arrow[r, dotted] \arrow[d] & \ker(h) \arrow[d] \arrow[dddll, dotted] & \\
 & A' \arrow[r] \arrow[d, "f"'] & B' \arrow[r] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & \\
 & \operatorname{coker}(f) \arrow[r, dotted] \arrow[d] & \operatorname{coker}(g) \arrow[r, dotted] \arrow[d] & \operatorname{coker}(k) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Definition: de Rham Complex

Let M be a smooth manifold. The de Rham complex of M is a sequence of spaces of differential forms:

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

Exactness at Level k

The de Rham complex is **exact at level** k if:

$$0 \to \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to 0$$

This means that every $(k+1)$-form in $\Omega^{k+1}(M)$ is the exact differential of some k-form in $\Omega^k(M)$.

Applications of Abelian Categories

- Algebraic Topology and Homological Algebra
- Representation Theory
- Sheaf Theory and Algebraic Geometry
- Derived Functors

Example

The Koszul complex used in algebraic geometry and commutative algebra to study properties of rings and modules.